A localization of Γ-measurability

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Abstract

We introduce the notion of Γ continuity at a point x—where Γ is any pointclass—and give conditions under which Γ continuity at every x is equivalent to Γ measurability. Using this we extend the notion of the integral of a measurable function. Also we examine the case Γ = Σ0^1 ξ , where (Σ0^1 ξ )ξ<ω1 is the usual ramification of the class of Borel sets, see [A.S. Kechris, Classical Descriptive Set Theory, Springer-Verlag, 1994].

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1. Introduction

With the letters X and Y we always mean metric spaces, unless stated otherwise. Also with the term “pointclass” we mean a collection of sets in arbitrary spaces, for example, the class of open sets. If Γ is a pointclass and X is a metric space we denote with Γ(X) the family of the subsets of X which are in Γ.

Recall that if Γ is an arbitrary pointclass and f is a function from X to Y, we call f Γ measurable iff for each open G ⊆ Y the inverse image f⁻¹[G] is also in Γ [1].

We will consider some special cases for Γ. First define the family Σ^0_1(X) as the collection of all open subsets of X. Put also Π^0_1(X) = {A ⊆ X/X \ A ∈ Σ^0_1(X)}, i.e. the family Π^0_1(X) is the collection of closed subsets of X.

By transfinite recursion we define for each ξ < ω1 [1]

\[ \Sigma^0_\xi(X) = \bigcup_{n \in \omega} A_n / \text{where } A_n \in \Pi^0_{\xi_n}(X) \text{ for some } \xi_n < \xi, \forall n \in \omega \]

and

\[ \Pi^0_\xi(X) = \{ A \subseteq X/X \setminus A \in \Sigma^0_\xi(X) \}. \]

Put also Δ^0_\xi(X) = Σ^0_\xi(X) ∩ Π^0_\xi(X), for each ξ < ω1.

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It is well known that \( \Delta^0_\xi(X) \subseteq \Sigma^0_\xi(X) \subseteq \Delta^0_{\xi+1}(X) \) for each \( \xi < \omega_1 \) and that \( \bigcup_{\xi < \omega_1} \Sigma^0_\xi(X) = \bigcup_{\xi < \omega_1} \Pi^0_\xi(X) = \mathcal{B}(X) \), where \( \mathcal{B}(X) \) is the Borel \( \sigma \)-algebra on \( X \).

Recall that a topological space \( X \) is called Polish iff it is separable and metrizable by some metric \( d \) such that \((X, d)\) is complete. If \( X \) is a perfect Polish space (i.e. a Polish space with no isolated points), then for each \( \eta < \xi < \omega_1 \) there exists a set \( A \subseteq \Delta^0_\xi(X) \setminus \Delta^0_\eta(X) \). With \( \Sigma^0_\xi \) we mean the class of all sets which belong to \( \Sigma^0_\xi(X) \) for some \( X \).

2. \( \Gamma \) continuity

We now give a notion which is closely connected to \( \Gamma \) measurability.

**Definition 2.1.** Let \( \Gamma \) be an arbitrary class of sets, a function \( f : X \to Y \) and \( x \in X \). The function \( f \) is called \( \Gamma \) continuous at \( x \) iff for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that the set \( f^{-1}[S(f(x), \varepsilon)] \cap S(x, \delta) \) is in \( \Gamma \).

Also say that \( f \) is \( \Gamma \) continuous at every \( x \in X \).

(Of course with \( S(f(x), \varepsilon) \) we mean the set \( \{ y \in Y : p(f(x), y) < \varepsilon \} \) where \( p \) is the metric of \( Y \). The same for \( S(x, \delta) \).)

We will denote the set \( f^{-1}[S(f(x), \varepsilon)] \cap S(x, \delta) \) with \( A(x, \varepsilon, \delta) \) or simpler with \( A(\varepsilon, \delta) \).

We first concentrate on the case where \( \Gamma = \Sigma^0_\xi \) for some \( \xi < \omega_1 \). The following are easy consequences of the definitions.

**Remark 2.2.**

(I) Assume that \( \Gamma \) contains the class of open sets and is closed under finite intersections. If \( f : X \to Y \) is \( \Gamma \) measurable then \( f \) is \( \Gamma \) continuous.

(II) A function \( f : X \to Y \) is \( \Sigma^0_1 \) continuous exactly when it is continuous, or equivalently iff it is \( \Sigma^0_0 \) measurable.

(III) Let \( \Sigma^0_\xi \) measurable functions \( f, g : X \to \mathbb{R} \) and \( \lambda \in \mathbb{R} \). Then the functions \( f + g, \lambda \cdot f, f \cdot g, |f|, \max\{f, g\} \) and \( \min\{f, g\} \) are also \( \Sigma^0_\xi \) measurable.

(IV) If \( \eta < \xi < \omega_1 \) and \( f : X \to Y \) is \( \Sigma^0_\eta \) measurable (or \( \Sigma^0_\xi \) continuous at some \( x \in X \)), then \( f \) is also \( \Sigma^0_\xi \) measurable (respectively, \( \Sigma^0_\xi \) continuous at \( x \)).

**Example 2.3.**

(I) Define \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
    f(x) = \begin{cases} 
        0, & \text{if } x \leq 0, \\
        1, & \text{if } x > 0. 
    \end{cases}
\]

For each \( \varepsilon > 0 \) put \( \delta = 1 > 0 \). Then \( A(0, \varepsilon, 1) = (x - 1, 0] \) for each \( x \in \mathbb{R} \). Hence \( A(0, \varepsilon, 1) \in \Sigma^0_2(\mathbb{R}) \). Therefore \( f \) is \( \Sigma^0_2 \) continuous at \( 0 \), but not continuous at \( 0 \).

(II) Let \( f : \mathbb{R} \to \mathbb{R} \) be the Dirichlet function, i.e.

\[
    f(x) = \begin{cases} 
        1, & \text{if } x \in \mathbb{Q}, \\
        0, & \text{if } x \notin \mathbb{Q}. 
    \end{cases}
\]

For \( x \in \mathbb{Q} \) and \( \varepsilon > 0 \) the set \( A(x, \varepsilon, 1) \) is one of the sets \((x - 1, x + 1) \cap \mathbb{Q}, \mathbb{R} \). In either case \( A(x, \varepsilon, 1) \) is an \( F_\sigma \) set and therefore \( f \) is \( \Sigma^0_2 \)-continuous on every rational \( x \). Also for each rational \( x \) and for each \( \delta > 0 \) the set \( A(x, \frac{1}{2}, \delta) = (x - \delta, x + \delta) \cap \mathbb{Q} \) is not a \( G_\delta \) set. Hence \( f \) is not \( \Pi^0_2 \)-continuous at \( \mathbb{Q} \).

With analogous arguments the Dirichlet function is \( \Pi^0_2 \) but not \( \Sigma^0_2 \) continuous at every \( x \notin \mathbb{Q} \).

It is interesting to distinguish \( \Sigma^0_\xi \) continuity from \( \Sigma^0_\eta \) continuity. This is fairly easy to do at this point. However, we will obtain this as a result of Theorem 2.5.

According to Remark 2.2, \( \Sigma^0_\xi \) measurability implies \( \Sigma^0_\xi \) continuity. Of course the inverse is also true in the case of \( \xi = 1 \). This is because the class of \( \Sigma^0_1 \) sets is closed under arbitrary unions. However the latter does not hold.
for the general case; so we cannot refer to an analogous proof in order to establish the inverse in the case of an arbitrary $\xi$.

On the other hand, in some topological spaces we can replace arbitrary unions of open sets with countable ones, i.e. those spaces which satisfy the Lindelöf property. It is well known that separable metric spaces are among those spaces. This remark will help us prove that $\Sigma^0_\xi$ measurability is equivalent to $\Sigma^0_\xi$ continuity in separable metric spaces.

**Lemma 2.4.** Let $(X,d)$ and $(Y,p)$ be separable metric spaces and a function $f : X \to Y$. For each open $G \subseteq Y$ which is bounded (i.e. $\sup_{x,y \in G} p(x,y) < \infty$) there exists a family $(\varepsilon_n)_{n \in f^{-1}[G]}$ of positive reals, such that for every family $(\delta_n)_{n \in f^{-1}[G]}$ of positive reals there exists a countable interval $I \subseteq f^{-1}[G]$ with

$$f^{-1}[G] = \bigcup_{x \in I} A(x, \varepsilon_n, \delta_n)$$

(where $A(x, \varepsilon_n, \delta_n) = f^{-1}[S(f(x), \varepsilon_n)] \cap S(x, \delta_n)$).

Roughly speaking this lemma transfers arbitrarily large unions in which we may concern, into countable unions.

**Proof.** Let $G \subseteq Y$ open and bounded. Then for each $x \in f^{-1}[G]$ there exists $r > 0$ such that $S(f(x), r) \subseteq G$. Put $\varepsilon_n = \sup \{ r > 0 : S(f(x), r) \subseteq G \} > 0$. Then $\varepsilon_n \in \mathbb{R}$ since $G$ is bounded.

One can check that $S(f(x), \varepsilon_n) \subseteq G$ for each $x \in f^{-1}[G]$.

Take $(\delta_n)_{n \in f^{-1}[G]}$ any family of positive reals.

Put $A = \{ S(f(x), \varepsilon_n) / x \in f^{-1}[G] \}$. From the Lindelöf property of $Y$ there exists a sequence $(x_n)_{n \in \omega}$ of elements of $f^{-1}[G]$ such that $\bigcup_{n \in \omega} S(f(x_n), \varepsilon_n) = \bigcup A$.

For each $n \in \omega$ put $B_n = \{ x \in X / f(x) \in S(f(x_n), \varepsilon_n) \} \subseteq f^{-1}[G]$ and $B_n^* = \{ x \in X / f(x) \in B_n \}$. From the Lindelöf property of $X$ there exists a sequence $(\delta_n^*)_{n \in \omega}$ of elements of $B_n^*$ such that $\bigcup_{n \in \omega} S(x_n, \delta_n^*) = \bigcup_{n \in \omega} B_n^*$, for each $n \in \omega$.

Define $I = \{ x_n^k / k \in \omega \}$. Observe that if $x = x_n^k$ for some $n, k \in \omega$, then from the definition of $\varepsilon_n$ it follows that $\varepsilon_n \leq 2 \varepsilon_n$. Using this remark it is easy to verify that

$$f^{-1}[G] = \bigcup_{x \in I} A(x, \varepsilon_n, \delta_n).$$

\[\square\]

**Theorem 2.5.** Let $X,Y$ be separable metric spaces and a function $f : X \to Y$.

(I) If there exists a function $\xi : X \to \omega_1$ such that for each $x \in X$ the function $f$ is $\Sigma^0_\xi$ continuous at $x$, then $f$ is $\Sigma^0_\xi$ measurable for some countable ordinal $\xi \leq \sup_{x \in X} \xi(x)$.

(II) For each $\xi < \omega_1$, $f$ is $\Sigma^0_\xi$ measurable exactly when it is $\Sigma^0_\xi$ continuous at every $x \in X$.

**Proof.** It is enough to prove (I) since the second assertion follows immediately from the first.

Let $\{ G_n / n \in \omega \}$ be a countable basis for the topology of $Y$. Since $Y$ satisfies the Lindelöf property we may assume that each $G_n$ is an open ball in $\mathbb{R}$ and hence bounded (with respect to the metric of $Y$).

For each $n \in \omega$ choose a family of positive reals $(\varepsilon_n^1, \varepsilon_n^2, \varepsilon_n^3, \varepsilon_n^4)_{x \in f^{-1}[G_n]}$ as in Lemma 2.4.

Now fix some $n \in \omega$. For each $x \in f^{-1}[G_n]$, $f$ is $\Sigma^0_\xi$ continuous at $x$. So for $\varepsilon_n^1 > 0$, there exists $\delta_n > 0$ with $A(x, \varepsilon_n^1, \delta_n) \in \Sigma^0_\xi(x)$. Choose a family $(\delta_n)_{x \in f^{-1}[G_n]}$ of those $\delta$’s. From Lemma 2.4 there exists a countable set $I \equiv I_n \subseteq f^{-1}[G_n]$ such that

$$f^{-1}[G_n] = \bigcup_{x \in I_n} A(x, \varepsilon_n^1, \delta_n).$$

If we let $\zeta_n = \sup_{x \in I_n} \xi(x)$ then $\zeta_n$ is a countable ordinal and $A(x, \varepsilon_n^1, \delta_n) \in \Sigma^0_{\zeta_n}$ for each $x \in I_n$. So $f^{-1}[G_n] \in \Sigma^0_{\zeta_n}$.

Putting $\xi = \sup_{n \in \omega} \zeta_n$, where $\zeta_n$ is as above, we obtain the result. \[\square\]
The preceding theorem allows to view—in separable metric spaces—the notion of \( \Sigma^0_\xi \) measurability in a “local way”, since \( \Sigma^0_\xi \) continuity is a local meaning.

This proof applies also to other pointclasses. Let us first recall the class of sets with the Baire property. If \( A \) is a subset of a topological space \( X \), we say that \( A \) has the Baire property if there exists some open \( U \) such that the symmetric difference \( (A \setminus U) \cup (U \setminus A) \) is meager in \( X \).

We denote the class of the sets with the Baire property with \( BP \). It is well known that for any topological space \( X \), the family \( BP(X) \) forms the least \( \sigma \)-algebra on \( X \) which contains all open and all meager sets.

**Corollary 2.6.** Let \( X, Y \) be separable metric spaces and a function \( f : X \to Y \). Also let a pointclass \( \Gamma \) which contains the open sets and it is closed under countable unions and finite intersections.

Then \( f \) is \( \Gamma \) measurable exactly when it is \( \Gamma \) continuous at every \( x \in X \).

In particular \( f \) is \( BP \) measurable, i.e. Baire measurable (or \( \Sigma^0_\xi \) measurable) exactly when it is \( BP \) continuous at every \( x \in X \) (respectively, \( \Sigma^0_\xi \) continuous at every \( x \in X \)).

**Proof.** Proceed as in the proof of Theorem 2.5. \( \square \)

We now give an example distinguishing \( \Sigma^0_\xi \) continuity from \( \Sigma^0_\eta \) continuity, for each \( \eta < \xi < \omega_1 \), using Theorem 2.5.

**Example 2.7.** Let \( X \) be a perfect Polish space, \( \eta < \xi < \omega_1 \) and \( A \in \Delta^0_\xi(X) \) but not in \( \Delta^0_\eta(X) \).

Then the characteristic function of \( A, \chi_A \) is easily \( \Sigma^0_\xi \) measurable (and hence \( \Sigma^0_\xi \) continuous) but not \( \Sigma^0_\eta \) measurable. From the previous theorem there exists some \( x \in X \) such that \( f \) is not \( \Sigma^0_\eta \) continuous at \( x \). Therefore \( f \) is \( \Sigma^0_\xi \) continuous but not \( \Sigma^0_\eta \) continuous.

We conclude with an application of Corollary 2.6. Let us begin with some notations. If \( X \) is a metric space, \( A \) is a subset of \( X \) and \( \Gamma \) is a pointclass denote with \( \Gamma_A \) the family \( \{ G \cap A / G \) is a subset of \( X \) and in \( \Gamma \} \). Also for a function \( f : X \to Y \) put \( X(f, \Gamma) = \{ x \in X / \) the function \( f \) is \( \Gamma \) continuous at \( x \} \). Finally for \( A \subseteq X \) denote with \( \chi_A \) the characteristic function of \( A \).

Let \( X \) be a separable metric space and a function \( f : X \to \mathbb{R} \). We will consider the case where \( \Gamma \) is a \( \sigma \)-algebra \( \mathcal{M} \) which contains the open sets. Also let some \( A \subseteq X(f, \mathcal{M}) \equiv X_f \). Then for each \( x \in A \) the function \( f \) is \( \mathcal{M} \) continuous at \( x \). It is clear that the function \( f|A \) (i.e. the restriction of \( f \) on \( A \)), is \( \mathcal{M}_A \) continuous at every \( x \in A \). Now regard \( A \) as a metric space. From Corollary 2.6 it follows that the function \( f|A \) is \( \mathcal{M}_A \) measurable. If furthermore \( A \in \mathcal{M} \) one can verify that the function \( f : \chi_A \) is \( \mathcal{M} \) measurable.

So if \( \mu \) is a measure on \((X, \mathcal{M})\) the integral \( \int_A f \, d\mu \) is well defined for each \( A \in \mathcal{M} \) with \( A \subseteq X_f \).

**Definition 2.8.** Let \( X \) be a separable metric space, \( \mathcal{M} \) be a \( \sigma \)-algebra on \( X \) which contains the open sets and \( \mu \) is a measure on \((X, \mathcal{M})\). Let also a non-negative function \( f : X \to \mathbb{R} \). Define the **partial integral** of the function \( f \) as follows:

\[
\int_A^P f \, d\mu = \sup \left\{ \int_A f \, d\mu : A \in \mathcal{M} \land A \subseteq X_f \right\}.
\]

**Remark 2.9.** Notice that if \( X_f \) is in \( \mathcal{M} \) since \( f \) is non-negative we have that \( \int_X f \, d\mu = \int_{X_f} f \, d\mu \). The set \( X_f \) is sort of speak the “largest” set on which the function \( f \) behaves like a measurable function. It would be interesting to find conditions under which the set \( X_f \) is in \( \mathcal{M} \).

If furthermore \( X_f = X \) then the partial integral coincides with the usual integral. Thus this new notion is indeed a generalization of the classic one.

Of course we may extend the previous notion to a not necessarily non-negative function with the usual way. For an arbitrary function \( f : X \to \mathbb{R} \) let \( \int_A^P f \, d\mu = \int_A^P f^+ \, d\mu - \int_A^P f^- \, d\mu \), where \( f^+ = \max\{f, 0\} \) and \( f^- = \max\{-f, 0\} \). (In case of \( \infty - \infty \) we define the partial integral to be 0.)
It would be interesting also to examine if the classic theorems of the usual integral can be transferred to the partial integral, see [2].

Example 2.10. Here we give an example of a partial integral which is reduced to the Lebesgue integral on a Cantor-type set. Define \( x \sim y \Leftrightarrow x - y \in \mathbb{Q} \) and (using the Axiom of Choice) let \( A \) be a set which contains exactly one member of each equivalence class. Let \( M \) be the Lebesgue \( \sigma \)-algebra on \( \mathbb{R} \) and \( \mu \) be the Lebesgue measure on \((\mathbb{R}, M)\). It is well known that \( A \notin M \), in fact for every set \( U \subset \mathbb{R} \), if \( U \cap A \in M \) then \( \mu(U \cap A) = 0 \). This is because if \( \mu(U \cap A) > 0 \) then from the Steinhaus theorem there exists \( \delta > 0 \) such that \( (\delta, \delta) \subseteq (U \cap A) \sim (U \cap A) \subseteq A - A \), a contradiction.

Let \( \mathcal{B} \) be the family of all open intervals \( (x - \delta, x + \delta) \) for which \( A^c \cap (x - \delta, x + \delta) \in M \), where \( A^c \) stands for the complement of \( A \) in \( \mathbb{R} \). If \( \mathcal{B} = \emptyset \) put \( J = \emptyset \), otherwise choose a countable family \( (I_n)_{n \in \omega} \) of the previous intervals which covers \( \mathcal{B} \) and put \( J = \bigcup_{n \in \omega} I_n \). In any case we have that \( A^c \cap J \in M \) and if \( A^c \cap (x - \delta, x + \delta) \in M \) then \((x - \delta, x + \delta) \subseteq J \).

Define \( X = \mathbb{R} \setminus J \). Then \( X \) is a closed set and furthermore \( \mu(X) > 0 \). Otherwise the set \( A^c \cap J \) would \( \mu \)-null and thus in \( M \). Hence \( A^c = (A^c \cap J) \cup (A^c \cap J^c) \in M \), a contradiction. It is well known that we can find a Cantor type set \( C \subseteq \mathbb{R} \) for which \( \mu(C) > 0 \).

Let \( M_X \) be the restriction of \( M \) on \( X \), i.e. \( M_X = \{ M \cap X / M \in M \} \). Also let \( \mu_X \) be the restriction of \( \mu \) on \( M_X \).

Since \( X \) is closed and thus in \( M \) it is clear that some \( B \subseteq X \) is in \( M_X \) if and only if \( B \in M \).

Define the function \( g : X \to \mathbb{R} \) by \( g(x) = \inf \{|x - y|/y \in C\} \). The function \( \Gamma \) is continuous non-negative and \( g^{-1}([0]) = \{C\} \).

Now define \( f : X \to \mathbb{R} \) as follows
\[
f(x) = \begin{cases} 
g(x) + 1, & \text{if } x \in A, \\
1, & \text{if } x \notin A.
\end{cases}
\]

If \( x \in C \), i.e. \( g(x) + 1 = 1 \), it is clear that \( f \) is continuous (and hence \( M_X \) continuous) at \( x \). For simplicity put \( X_f = X(f, M_X) \). Let now some \( x \in X_f \setminus C \). Assume furthermore that \( x \in A \). Then \( g(x) > 0 \) and \( f(x) = g(x) + 1 \). Put \( \varepsilon = \frac{g(x)}{2} \). Notice that if \( y \notin A \) then \( f(y) = 1 < g(x) + 1 - \varepsilon = g(x) - \varepsilon \). Hence if \( f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon) \) then \( y \notin A \). It follows that
\[
\begin{align*}
f^{-1}[\{(f(x) - \varepsilon, f(x) + \varepsilon)\}] &= f^{-1}[\{(f(x) - \varepsilon, f(x) + \varepsilon)\}] \cap A \\
&= h^{-1}[\{(h(x) - \varepsilon, h(x) + \varepsilon)\}] \cap A,
\end{align*}
\]
where \( h = g + 1 \).

Since \( x \in X_f \) there exists some \( \delta > 0 \) such that \( f^{-1}[\{(f(x) - \varepsilon, f(x) + \varepsilon)\}] \cap (x - \delta, x + \delta) \in M_X \subseteq M \). Also we have that
\[
\begin{align*}
f^{-1}[\{(f(x) - \varepsilon, f(x) + \varepsilon)\}] \cap (x - \delta, x + \delta) \\
&= h^{-1}[\{(h(x) - \varepsilon, h(x) + \varepsilon)\}] \cap A \cap (x - \delta, x + \delta) \\
&= U_x \cap A,
\end{align*}
\]
where \( U_x \) is open in \( X \) such that \( x \in U_x \). Therefore \( U_x \cap A \in M_X \subseteq M \) and thus \( \mu(U_x \cap A) = 0 \). Repeat the same procedure for each \( x \in (X_f \setminus C) \cap A \) in order to get the previous set \( U_x \). Using the Lindelöf property of \( X \) we find a sequence \( (x_n)_{n \in \omega} \) in \((X_f \setminus C) \cap A \) such that \( (X_f \setminus C) \cap A = \bigcup_{n \in \omega} U_{x_n} \cap A \). It follows that the set \( (X_f \setminus C) \cap A \) is \( \mu \)-null and thus it belongs in \( M \).

Now towards a contradiction assume that there exists some \( x \in X_f \setminus C \) which is not in \( A \). Then \( f(x) = 1 \) and \( g(x) > 0 \). Put \( \varepsilon = \frac{g(x)}{2} > 0 \). Since \( g \) is continuous there exists some \( \delta_0 > 0 \) such that for all \( y \in (x - \delta_0, x + \delta_0) \cap X \) we have that \( g(y) > \frac{g(x)}{2} \).

Also \( x \in X_f \) hence there exists some \( \delta > 0 \) such that \( f^{-1}[\{(f(x) - \varepsilon, f(x) + \varepsilon)\}] \cap (x - \delta, x + \delta) \in M_X \subseteq M \). We may assume that \( \delta \leq \delta_0 \). Thus if \( y \in (x - \delta, x + \delta) \cap X \) then \( f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon) \Leftrightarrow y \notin A \). It follows that \( f^{-1}[\{(f(x) - \varepsilon, f(x) + \varepsilon)\}] \cap (x - \delta, x + \delta) = A^c \cap (x - \delta, x + \delta) \cap X = A^c \cap J^c \cap (x - \delta, x + \delta) \in M \).

Since \( A^c \cap J \in M \) it follows that \( A^c \cap J \cap (x - \delta, x + \delta) = [A^c \cap J^c \cap (x - \delta, x + \delta)] \cup [A^c \cap J \cap (x - \delta, x + \delta)] \in M \). From the definition of \( J \) it follows that \( (x - \delta, x + \delta) \subseteq J \). This is a contradiction since \( x \in X = \mathbb{R} \setminus J \).
Therefore $X_f \setminus C = (X_f \setminus C) \cap A$ is $\mu$-null and thus in $\mathcal{M}$. Since $X_f \setminus C \subseteq X$ we have that $X_f \setminus C \in \mathcal{M}_X$.

Furthermore $C$ is closed in $X$ and it is also a subset of $X_f$, hence $X_f \in \mathcal{M}_X$.

Thus $\int f \, d\mu_X = \int_{X_f} f \, d\mu_X = \int_C f \, d\mu_X = \int_C 1 \, d\mu_X = \mu_X(C) = \mu(C) > 0$.

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